TIFR CENTRE for Applicable Mathematics

A Vision Document

Mathematics has been, and continues to be, the most basic of sciences, and a key tool in engineering. The extent of application of Mathematics to the real world issues has increased manifold in recent decades. Basic research in physical sciences has always hinged on new mathematical ideas, since the early days. The impact of mathematics is now being strongly felt in the life sciences and medicine as well. Similarly, the massive strides in information technology have led to an increasing use of mathematics in industry. The new generation of problems coming from Industry and the life sciences have posed several interesting challenges for applied mathematicians, paving the way for them to contribute directly to progress in these areas.

In the Indian context, the rapid growth in various sectors of Indian industry, the great achievements in space and atomic energy research and the emergence of biotechnology, financial services and other knowledge based industries forces us to seek and develop a strong mathematical base which can contribute to this revolution. In this context it is imperative to set up a full-fledged centre for Applicable mathematics, aimed at meeting the new-age requirements.

Strategic Objectives of the Centre

In the light of the historical experience around the world we see that such a centre should have before it the following objectives.

1. To pursue high quality research in the various areas identified as applicable mathematics, and their applications.
2. To pro-actively interact with other scientists, engineers, other potential users of mathematics, and those who need creative solutions involving state of art ideas in the applications of mathematics in their fields of activity.
3. To provide education and training at various levels, aimed at building up expertise, and to disseminate the know-how for practical use.

Sometimes there is an inclination to interpret the term applicable mathematics to mean pure theory, as against applied mathematics which is taken to mean involving in direct applications. There is a need to transcend such a divide of interpretation. Experience shows that focussing only on problems of direct application limits the perspective and hampers long term development. On account of this general sense the term applicable mathematics is preferable on account of its wider connotation. It should of course be understood nevertheless that to look for possibilities of applications on a continuing basis has to be an integral part of such an activity.

Thrust areas

We propose that the following as the thrust areas to be developed at the centre, in the coming years. The areas are chosen taking into account on the one hand the needs in the context of the overall developments worldwide, and on the other hand the resources and strengths as at present on the basis of which we can realistically aim to develop them.

1. Computational Methods for Hyperbolic equations
2. Image Processing and Interpretation
3. Modelling of Atmospheric phenomena
4. Numerical methods for control and optimization of industrial processes
5. *Mathematical Finance*
6. *Mathematical issues in Medicine and Biology.*

Mobilisation

At our Centre at present considerable expertise has built up, through concerted efforts, in various areas of applications of mathematics. In particular, the members of the center have contributed immensely to the better understanding of linear and nonlinear partial differential equations of both of the elliptic as well as hyperbolic type. The areas of application that we focus on currently are in two-phase flow in porous media used in Petroleum reservoir simulation, waste water treatment plants and other mineral process, issues in image processing, problems in atmospheric and ocean sciences, and homogenization and control of differential equations.

Thus in many topics mentioned above, to quite an extent the necessary leadership to achieve the aims can come from the Centre itself. It would be beneficial, however, to bring in also other suitable persons in the relevant areas to be part of the activity, so that the development towards achieving the aims are accelerated. It would be good if we can attract some talented applied mathematicians to join us on regular positions so that the potential of the Centre is substantially improved. We are on the look out for possible candidates for the purpose. In the interim we propose that more post-doctoral fellows may be appointed in the relevant areas to give a boost to the activity; some of the post-doctoral fellows may be absorbed depending on their performance.

Recently we have proposed the names of Prof. J. Bona and Prof. S. Chanillo for the post of adjunct faculty. We expect them to play a significant role in achieving some of our goals.

With regard to the area of *applications in Physical and Engineering Sciences* we plan to proceed in two ways, to increase the interaction. A post doctoral fellowship has been offered at the Centre to an Engineer and we will pursue with this idea (we are also open to the idea of absorbing depending on their suitability). Efforts are also on to get Engineers from other institutions at the Centre for interaction. It may be mentioned in this respect that nearly half of our colloquium talks over the last 5-6 years were by Engineers; this has led to a very useful collaboration in some cases and such interaction is expected to recur.

In certain areas which involve expertise in geometry (areas like shape optimization) we expect to draw support from colleagues in the School of Mathematics, Mumbai. On some aspects of computational Mathematics we hope to utilize the expertise of Prof. N. Karmarkar. Some discussion with him have taken place and we expect to have a fruitful interaction with his group.

Organization like BARC, DRDO and ISRO have the necessary appreciation for the kind of Mathematics vision we have presented. However in order to have impact which will be mutually beneficial it would be convenient if scientists from these institutions visit the Centre for periods ranging from 3 months to a year. We propose to approach the institutions suggesting that they depute some of their scientists accordingly. We would also welcome possibilities of mathematicians from the Centre visiting these institutions for suitable durations to interact or to provide expertise.

On Biology related mathematics we look forward to having a collaboration with the **National Centre for Biological Sciences,** they also being interested in developing this area.

Financial Mathematics is growing in its popularity, and efforts are on to recruit experts of high standard in the field of stochastic differential equations to develop a group in the area.

On the student training front, the recent Deemed University Status of TIFR has been a very beneficial development. We are initiating doctoral and integrated Ph.D. programmes at the Centre. It is expected that in a few years this would provide good strength to the Centre at the younger level.

Research Perspectives

In this section we discuss in some detail the perspectives on research problems to be pursued.

During the last decades numerical simulations have become an in dispensable tool in Science and Engineering. Partial Differential Equations and their numerical solutions are an important ingredient in the design of such programmes. Even if the basic theory is largely understood there are challenging problems, especially non-linear problems, singular perturbations problems etc.

Study of singular perturbation problems and relevant numerical methods is a challenging and important research area in PDEs. Construction of numerical methods for PDEs depends very much on the kind of properties the underlying equation possesses. However, for singular perturbation problem the basic type of equation changes as the perturbation parameters reach critical values and this creates basic difficulties. Typical prototypes are nonlinear convection-diffusion equations. Applications abound when one considers multiphase flow in porous media.

Degenerate convection-diffusion equation (where diffusion is degenerating) is a class of nonlinear PDEs are difficult to analyse theoretically, therefore numerical solutions are a tremendous help. However, the numerical schemes are largely ad hoc. In this context numerical schemes in an entropy frame work seems to be very promising and such an idea has been proposed by members at this centre, this has been well received by the mathematical community and we hope to built on this strength. Conservation Laws and Hamilton-Jacobi equations with discontinuous coefficients is another area where the numerical methods based on entropy frame work promises to yield very fruitful results and these have important applications in the oil industry and the problem of shape from shading in image processing. Again there is significant contribution from this centre.

The construction of numerical methods for constrained evolution equations represent a challenging problem in numerical methods for PDEs. The most important of such equations are the Navier-Stokes equation and Maxwell's equations. There are others like the Yang-Mills equations. However given the strength of the Centre as of now and the importance of Navier-Stokes equations in real world applications, concentrated efforts will be focused on this equation. Also it is worth mentioning the regularity of the solution for N-S is one of the important challenge facing the mathematical community (This is one of the proposed Millennium prize problem by Clay Math Organization). It planned that the Centre will arrange a 6 month special programme devoted to the study of N-S both numerical and theoretical within the next 1 to 2 years. We hope to create a group which will concentrate on the study of N-S equation comprising of all aspects.

One of the important studies carried out at this Centre which is in the realm of intersection of applications in the Physical Sciences and Mathematical Analysis of Differential equations is in the area of Atmospheric radiation. Such a work has been carried out in collaboration with Scientists at the Centre for Atmospheric Sciences and Aerospace Engineering of IISc. Two success stories in this regard has been the satisfactory explanation of the Ramdas layer and developing a one dimensional radiation code that is being used in another code that studies the Indian Monsoon variability. The next step is to develop a three dimensional radiation code that could be used for the same purpose as mentioned above. Developing this code would be an extremely tedious task that would require not only powerful computers but also engineers with good mathematical training or mathematicians with an engineering inclination.

High dimensional computational problems are common these days due to its applications in finance and data mining. Although mathematical theory for infinite dimensional problems is well developed and efficient algorithms exist in dimensions up to three the same is not true for problems with dimensions exceeding (for instance) hundred. In other words efficient algorithms for finite but high dimension is a territory waiting to be explored and it is planned to have a group working on this.

The area of image processing and shape optimization is already being pursued at Centre by our research students and they will be strongly supported to continue working in this area by having experts visiting this Centre.

At the level of theoretical work we will continue to pursue the studies of non-linear PDEs, which has been one of our very strong points, through methods of variational calculus, bifurcation theory, homogenization techniques, optimal control etc. Another important topic to be pursued is the viscosity solutions of generalised Hamilton-Jacobi-Bellman (HJB) equations, an important ingredient in financial mathematics. Also the general study of hyperbolic equations in particular wave propagation and its applications, propagation of singularities etc.